Fall 2019

MATH3060 Mathematical Analysis III

Selected Solution to the Mid-Term Examination

Answer all five questions.

1. (25 points) Find the Fourier series of the function $f(x) = |x|$ (extended as a 2π-periodic function). Discuss its convergence.

2. (15 points) Let f be a continuous function on $(-\infty, \infty)$ which vanishes outside $[-1, 1]$. Define the function $\Phi(x, y)$ by

$$\Phi(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(x-t)^2+y^2} f(t)\, dt \ , \quad \forall (x,y),\ y > 0 \ .$$

Show that

$$\lim_{y\to 0^+} \Phi(x,y) = f(x) \ .$$

Solution. Using the formula

$$\int_0^{\infty} \frac{1}{1+z^2}\, dz = \frac{\pi}{2} \ ,$$

one shows that

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(x-t)^2+y^2}\, dt = 1 \ .$$

It follows that

$$\Phi(x,y) - f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(x-t)^2+y^2} (f(t) - f(x))\, dt \ .$$

For a fixed x, by the continuity of f, for $\varepsilon > 0$, there is some δ_0 such that $|f(t) - f(x)| < \varepsilon/2$ for all $t, |t - x| < \delta_0$. Therefore,

$$\left| \frac{1}{\pi}\int_{x-\delta_0}^{x+\delta_0} \frac{y}{(x-t)^2+y^2} (f(t) - f(x))\, dt \right| \leq \frac{\varepsilon}{2} \ . \qquad (1)$$

On the other hand, for $t \in (-\infty, x-\delta_0] \cup [x+\delta_0, \infty)$ we have $(x-t)^2 \geq \delta_0^2$. Letting $M = \sup |f|$, we have

$$\begin{aligned} \left| \frac{1}{\pi} \int_{x+\delta_0}^{\infty} \frac{y}{(x-t)^2+y^2} (f(t)-f(x))\, dt \right| &\leq \frac{2My}{\pi} \int_{x+\delta_0}^{\infty} \frac{1}{(x-t)^2+y^2}\, dt \\ &\leq \frac{2My}{\pi} \int_{x+\delta_0}^{\infty} \frac{1}{(x-t)^2}\, dt \\ &= \frac{2My}{\pi} \int_{\delta_0}^{\infty} \frac{1}{z^2}\, dz \\ &= \frac{2M}{\pi\delta_0}\, y\ . \end{aligned}$$

Similarly, we have

$$\left| \frac{1}{\pi} \int_{-\infty}^{x-\delta_0} \frac{y}{(x-t)^2+y^2} (f(t)-f(x))\, dt \right| \leq \frac{2M}{\pi\delta_0} y\ .$$

Now, taking $\delta < \dfrac{\pi\delta_0}{8M}$, then

$$\left| \frac{1}{\pi} \left(\int_{-\infty}^{x-\delta_0} + \int_{x+\delta_0}^{\infty} \right) \frac{y}{(x-t)^2+y^2} (f(t)-f(x))\, dt \right| < \frac{\varepsilon}{2}\ , \qquad (2)$$

for all $y, 0 < y < \delta$. The desired result follows from combining (1) and (2).

Note. Similar ideas were used in the proof of Theorem 1.5 in Chapter 1 as well as Problem 5 in Assignment 3.

3. Let $\sum_{-\infty}^{\infty} c_n e^{inx}$ be the Fourier series of the real-valued function f over $[-\pi, \pi]$.

   (a) (10 points) For an integrable function f on $[-\pi, \pi]$, show that

$$2\pi \sum_{-\infty}^{\infty} |c_n|^2 \leq \int_{-\pi}^{\pi} |f|^2\ .$$

(b) (10 points) Show that the equality sign in (a) holds when f is a continuous 2π-periodic function which is piecewise C^1.

Solution. (b) In view of (a) it suffices to show the inequality from the other direction holds. Since the given function is 2π-periodic, continuous, and piecewise C^1, its Fourier series converges uniformly to itself. Given $\varepsilon > 0$, there is some N such that

$$|f(x) - \sum_{k=-N}^{N} c_k e^{ikx}| < \varepsilon \ ,$$

for all $x \in [-\pi, \pi]$. We have

$$\begin{aligned} 0 &\le \int_{-\pi}^{\pi} f^2(x)\,dx - 2\pi \sum_{k=-N}^{N} |c_k|^2 \\ &= \int_{-\pi}^{\pi} (f(x) - \sum_{-N}^{N} c_k e^{ikx})\overline{(f(x) - \sum_{-N}^{N} c_k e^{ikx})}\,dx \\ &\le 2\pi\varepsilon^2 \ . \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{-\pi}^{\pi} f^2(x)\,dx &\le 2\pi \sum_{-N}^{N} |c_k|^2 + 2\pi\varepsilon^2 \\ &\le 2\pi \sum_{-\infty}^{\infty} |c_k|^2 + 2\pi\varepsilon^2 \ . \end{aligned}$$

The desired result comes by letting $\varepsilon \to 0$.

Note. The key point is the uniform convergence of the Fourier series to the function itself.

4. (a) (10 points) Show that the set $\{f \in C[a,b] : \ f(x) > 0, \ \forall x \in [a,b]\}$ is an open set under the supnorm.

(b) (10 points) Consider P, the set of all polynomials restricted on $[a, b]$. Determine the closure and the interior of P in $C[a, b]$ under the supnorm.

Solution. (b) The closure of P is $C[a, b]$. For, by Weierstrass Approximation Theorem, for any $f \in C[a, b]$, given $\varepsilon > 0$, there is a polynomial p such that $\|f - p\|_\infty < \varepsilon$. On the other hand, P has empty interior. For, consider the ball $B_r(p)$ where p is a polynomial and r is any positive number, the function $f = p + \varepsilon e^x$, $\varepsilon > 0$, which is not a polynomial, would belong to $B_r(p)$ for all sufficiently small ε.

5. (a) (10 points) Let E be a non-empty set in the metric space (X, d). Show that the function $d_E(x) = \inf\{d(x, y) : y \in E\}$ is continuous in (X, d).

   (b) (10 points) Take X to be $\mathbb{R}^n$ under the Euclidean metric. Show that the only sets which are open and closed are the empty set and $\mathbb{R}^n$ itself. Suggestion: Let E be a non-empty, open and closed set which is not $\mathbb{R}^n$, and draw a contradiction by considering d_E.

   Solution. (b) Let E be a closed and open proper, non-empty subset of $\mathbb{R}^n$. We will draw a contradiction assuming such set exists. First of all, pick a point lying outside E and consider the function $d_E(x_0)$, the distance from x_0 to E. Since $\mathbb{R}^n \setminus E$ is open, $d_E(x_0) > 0$. On the other hand, let $\{x_n\} \subset E$, $|x_n - x_0| \to d_E(x_0)$ as $n \to \infty$. From $|x_n| \le |x_0| + d_E(x_0)$ we see that $\{x_n\}$ is a bounded sequence, so it contains a convergent subsequence. By passing to a subsequence, we may assume $x_n \to z$ as $n \to \infty$. Then $d_E(x_0) = \lim_{n\to\infty} |x_n - x_0| = |x_0 - z|$, that is, z realizing the distance from x_0 to E. As E is closed $z \in E$. As E is open, we can find a ball $B_r(z) \subset E$ for some small $r > 0$. It is now clear that we can find some point $w \in B_r(z)$ such that $|w - x_0| < |z - x_0| = d_E(x_0)$, contradicting the definition of z. (Note that in the last step, one may take $x_0 = (0, 0, \cdots, 0)$ and $z = (a, 0, \cdots, 0), a > 0$. Then w can be taken to be $(a - r/2, 0, \cdots, 0)$.)